Application of Bernoulli's Equation to Buoyant Systems

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The various forms of Bernoulli's equation as customarily written express static pressure energy in terms of absolute pressure, thereby restricting their direct application to systems situated *in vacuo*. It is shown here that recognition (1) of the universal presence of a fluid outside all systems over which Bernoulli's equation is written, and (2) of the practical necessity of measuring pressure within such systems as gauge pressures relative to the pressure of the exterior fluid leads to a more general set of equations and to a concept of buoyant static pressure and potential energies, as opposed to strictly absolute values.

Failure properly to distinguish between these two types of energy quantities may result in error and confusion in the application of the various forms of Bernoulli's equation.

The terms in the Bernoulli equation as generally written in hydraulics are often referred to as *heads* since the units are commonly expressed as *feet*. This concept of fluid columns standing to various heights is valuable and appropriate in applications to the flow of incompressible fluids. However, when it is applied to compressible fluids considerable error may be introduced. It is the purpose of this paper to present a more rigorous concept of the terms in Bernoulli's equation which will be appropriate for all fluids under any conditions.

The integrated form of Bernoulli's equation

$$pv + \frac{V^2}{2g_c} + Z = \text{constant} \qquad (1)$$

is a consequence of Newton's laws of motion and was developed at least fifty years before Helmholtz unambiguously stated the First Law of Thermodynamics. The equation is, however, a restricted form of the First Law and can be derived from the general energy equation (*1*). An analysis of the units and character of the terms of the general energy equation will, therefore, be applicable to the Bernoulli equation.

The general energy equation is written, according to accepted practice, as

$$Z_1 + U_1 + p_1v_1 + V_1^2/2g_c + Q = Z_2 + U_2 + p_2v_2 + V_2^2/2g_c + Ws \qquad (2)$$

The terms of this equation are energy terms and must be expressed in equivalent units, i.e., foot pounds force per pound mass. This can be accomplished by writing (2) as

$$Z_1\left(\frac{g}{g_c}\right) + JU_1 + p_1v_1 + V_1^2/2g_c + JQ = Z_2\left(\frac{g}{g_c}\right) + JU_2 + p_2v_2 + V_2^2/2g_c + W_s \qquad (3)$$

where J is the mechanical equivalent of heat (778 ft. lb. force/B.t.u.), g is the local acceleration of gravity (ft./sec.^{2}), and g_c is the dimensional constant in Newton's law [32.1740 (lb. mass) (ft.)/(lb. force)(sec.^{2})]. Combining Equation (3) with the first and second law and introducing the degree of reversibility F gives, in consistent units,

$$\frac{g}{g_c}(Z_2 - Z_1) + \int_{p_1}^{p_2} v\,dp + \frac{(V_2^2 - V_1^2)}{2g_c} = -JF - W_s \qquad (4)$$

If this equation is multiplied by g_c/g, a form of Bernoulli's equation results which includes friction and in which the units of each term are simply feet.

$$(Z_2 - Z_1) + \left(\frac{g_c}{g}\right)\int_{p_1}^{p_2} v\,dp + \frac{(V_2^2 - V_1^2)}{2g} = -\left(\frac{g_c}{g}\right)JF - \left(\frac{g_c}{g}\right)W_s \qquad (5)$$

Equation (5) contains the ratio g_c/g, which relates Newton's gravitational constant g_c to the local acceleration of gravity. For certain applications this ratio is unity, and the equation is commonly used without regard to the magnitude of the ratio. For general application, however, the equation as written in (5) must be used.

For isothermal flow and small changes in absolute pressures, Equation (4) becomes

$$(g/g_c)(Z_2 - Z_1) + (p_2 - p_1)v_{avg} + \frac{(V_2^2 - V_1^2)}{2g_c} = -JF - W_s \qquad (6)$$

The symbol p in the foregoing equations represents absolute pressure, a quantity customarily evaluated by adding Bourdon gauge pressures or manometer readings in pounds force per square feet to the barometric pressure expressed in the same units. Since the difference of two absolute pressures thus evaluated is the same as the difference between the original gauge pressures or the difference indicated upon a manometer, the pressures conveniently used in practice are gauge pressures.

In the application of Equations (1) through (6) the almost universal presence outside the system of some buoyant fluid such as air is generally neglected. This practice produces an error, as the term Δp, the difference of absolute pressures, is no longer exactly equal to ΔP, the difference in gauge pressures. The error is small when incompressible fluids or liquids are involved or when systems over which the equation is written contain no elevation change. For compressible fluids such as gases or for cases where the system contains an elevation change, the error can become so large as to yield quite inaccurate and even absurd results. This is illustrated in the following example.

Illustration 1

Problem. A cylindrical brick chimney 108 ft. tall (measured from the center line of the breeching) and 7.5 ft. in diameter develops a draft of 0.600 in. of water at the base of the chimney (barometer normal, atmospheric temperature 62°F.) when the furnace it serves is fired at such a rate as to produce 1.715 moles/sec. of stack gas having an average molecular weight of 30.2 and an average temperature of 500°F.

It is necessary to double the steam-generating capacity by installation of a second identical furnace and boiler. Rather than build another chimney, it is proposed to conduct the gases from the second furnace into the breeching of the first by means of a Y connection, as illustrated in Figure 1, and by installing an induced-draft fan at the base of the chimney to compensate for the reduction in natural draft. What will be the theoretical power consumption of such a fan?

Solution. Writing the Bernoulli equation as commonly used between d and e of Figure 1 for the original furnace (no friction loss in the breeching assumed) gives

$$\left(\frac{g}{g_c}\right)\Delta Z + \frac{\Delta V^2}{2g_c} + v\Delta P = -JF - W_s$$

If $W_s = 0$ and it is assumed that $V_1 = V_2$ and that $g/g_c = 1.0$

$$\Delta Z + v\Delta P + JF = 0$$

or with

$$JF = \Delta H_f = \frac{2fLV^2}{g_cD},$$

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$$(Z_2 - Z_1) + \frac{(P_2 - P_1)}{\rho_{avg}} + \Delta H_f = 0$$

$$Z_2 = 108 \text{ ft.}$$

$$Z_1 = 0$$

$$P_2 = 0 \text{ gauge pressure}$$

$$P_1 = -\frac{0.600 \times 62.3}{12}$$

$$= -3.115 \text{ lb./sq. ft.}$$

$$\rho_{avg} = \frac{30.2}{359} \times \frac{492}{960}$$

$$= 0.0431 \text{ lb./cu. ft.}$$

$$\Delta H_f = -108 - \left(\frac{3.115}{0.0431}\right)$$

$$= -108 - 72.3 = \frac{2fLV^2}{g_c D}$$

$$= -180.3$$

$$f = -\frac{180.3 g_c D}{2LV^2}$$

$$= -\frac{180.3 \times 32.17 \times 7.5}{2 \times 108 \times (27.2)^2}$$

$$= -0.273,$$

which is absurd as f is negative.

DERIVATION OF MODIFIED EQUATIONS

The modification needed to make Bernoulli's equation as generally used in its various forms applicable under all conditions is derived as follows.

The level of the mercury well of a conveniently situated barometer may be considered as the datum of elevation and the static pressure of the atmosphere at this level as the datum of pressure; the absolute pressures within a system at elevations Z_1 and Z_2 may be expressed as

$$p_1 = P_1 + B - Z_1(g/g_c)\rho_0 \quad (7)$$

$$p_2 = P_2 + B - Z_2(g/g_c)\rho_0 \quad (8)$$

where ρ_0 is the average density of the outside fluid between the datum and elevations Z_1 and Z_2, B is the barometric pressure at the elevation datum, and P_1 and P_2 are the pressure differences between system fluid and outside fluid measured at the elevations Z_1 and Z_2. Representing the quantities $(P_1 + B)$ and $(P_2 + B)$ by the symbols p_1' and p_2' and substituting into Equation (3) gives, upon rearrangement,

$$Ju_1 + Z_1(g/g_c)(1 - \rho_0 v_1) + p_1'v_1 + V_1^2/2g_c + JQ$$
$$= Ju_2 + Z_2(g/g_c)(1 - \rho_0 v_2) + p_2'v_2 + V_2^2/2g_c + W_s \quad (9)$$

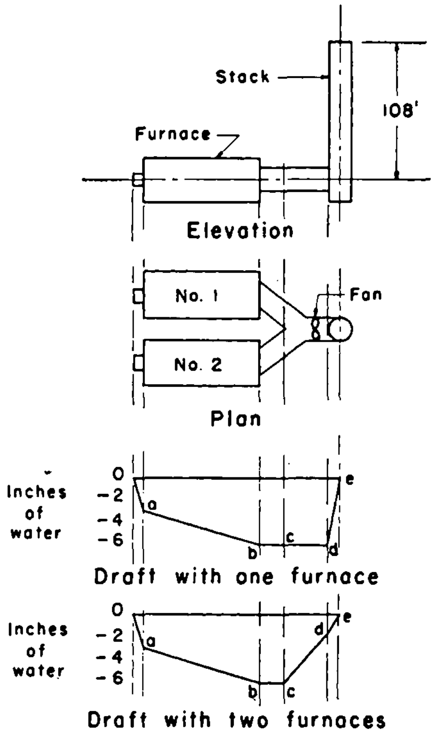

Fig. 1. Furnace arrangement.

From Equation (9) the corresponding mechanical-energy-balance equation is obtained as

$$Z_1(1 - \rho_0 v_1)(g/g_c) + V_1^2/2g_c - W_s - \int_{p_1'}^{p_2'} v\,dp - JF$$
$$= Z_2(1 - \rho_0 v_2)(g/g_c) + V_2^2/2g_c \quad (10)$$

If it is assumed that the variation of v is linear with Z, Equation (10) becomes, for isothermal flow and small changes in the absolute pressure,

$$(Z_2 - Z_1)(g/g_c)(1 - \rho_0 v_{Z_{avg}}) + (V_2^2 - V_1^2)/2g_c + (P_2 - P_1)v_{avg} = -JF - W_s \quad (11)$$

as $(p_2' - p_1')$ is identically $(P_2 - P_1)$.

The specific volume of the system fluid in the potential energy difference term has been written as $v_{Z_{avg}}$ to indicate that it is not necessarily the same quantity as v_{avg} in the static pressure energy term but is instead the arithmetic average value of v over only that portion of a system in which the elevation change from Z_1 to Z_2 takes place. Strictly speaking, of course, Bernoulli's equation when applied to gases should never be written over the entire length of a complicated system. The continual change in specific volume and velocity from point to point in a complex system makes it much safer to apply the theorem in its various forms in a series of steps, solving for intermediate values of pressure, density, and velocity before and after each important change in velocity, temperature, or elevation. Such a procedure, however, is quite laborious.

When the density ρ_0 of the outside fluid approaches zero, $(P_1 + B_1)$ and $(P_2 + B_2)$ approach p_1 and p_2, and the term $(Z_2 - Z_1)(g/g_c)(1 - \rho_0 v_{Z_{avg}})$ approaches $(Z_2 - Z_1)(g/g_c)$, showing that Equations (9), (10), and (11) are fundamentally the same as Equations (3), (4), and (6) but are directly applicable to any fluid under any conditions. The previous illustration can now be simply solved by use of Equation (11) and the convenient gauge pressure difference.

Applying Equation (11) to illustration 1, where it is assumed that $V_1 = V_2$, $W_s = 0$, and $g/g_c = 1.0$ as before

$$(Z_2 - Z_1)(1 - \rho_0 v_{Z_{avg}}) + \frac{\Delta P}{\rho_{avg}} + JF = 0$$

$$\rho_0 = \frac{29}{359} \times \frac{492}{522} = 0.0762 \text{ lb./cu. ft.}$$

$$V = \frac{1.715 \times 30.2}{0.0431} \times \frac{4}{(7.5)^2\pi}$$

$$= 27.2 \text{ ft./sec.}$$

$$(108 - 0)\left(1 - \frac{0.0762}{0.0431}\right) + \frac{(0 + 3.115)}{0.0431} + \Delta H_f = 0$$

$$-\Delta H_f = -\frac{2fLV^2}{g_c D}$$

$$= (108 - 191) + 72.3$$

or

$$\Delta H_f = 83 - 72.3 = 10.7$$

and

$$f = \frac{10.7 \times g_c \times D}{2LV^2}$$

$$= \frac{10.7 \times 32.17 \times 7.5}{2 \times 108 \times (27.2)^2}$$

$$f = 0.0162$$

Now

$$Re = \frac{Dv\rho}{\mu}$$

$$= \frac{7.5 \times 27.2 \times 0.0431}{0.028 \times 0.000672}$$

$$= 467{,}000$$

The roughness factor f/D, therefore, is 0.04 to 0.05, a condition where f is independent of Re. For the new furnace the velocity doubles but $f = 0.0162$ and the new draft at the base of the furnace is

$$-\frac{\Delta P}{\rho_{avg}} = \frac{P_1 - P_2}{\rho_{avg}}$$

$$= (Z_2 - Z_1)(1 - \rho_0 v_{Z_{avg}}) + \Delta H_f$$

$$\frac{P_1 - 0}{0.0431} = (108 - 0)(1 - \rho_0 v_{Z_{avg}})$$

$$+ 0.0162 \times \left(\frac{54.4}{27.2}\right)^2 \times 662$$

$$= -83 + 42.9 = -40.1$$

$$P_1 = -0.0431 \times 40.1$$

$$= -1.73 \text{ lb./sq. ft.}$$

Writing Equation (11) from c to d in Figure 1 gives

$$\frac{(\Delta V)^2}{2g_c} + v_{avg}\Delta P + W_s = 0$$

where $\Delta Z = 0$

$$-W_s = \frac{(V_d^2 - V_c^2)}{2g_c} + \frac{P_d - P_c}{\rho_{avg}}$$

$$= \frac{(54.4)^2 - (27.2)^2}{2 \times 32.17}$$

$$+ \frac{(-1.73 + 3.115)}{0.0431}$$

$$= 34.5 + 32.1$$

$$= 66.6 \frac{\text{ft. lb. force}}{\text{lb. mass}}$$

Therefore, the theoretical hp. =

$$\frac{66.6 \times 1.715 \times 30.2 \times 2}{550} = 12.6 \text{ hp.}$$

In transforming the sum of the static pressure and potential energy terms from the form

$$pv + Z(g/g_c) \quad (12)$$

$$= [B + P - Z(g/g_c)\rho_0]v + Z(g/g_c)$$

to

$$pv + Z(g/g_c) \quad (13)$$

$$= (B + P)v + Z(g/g_c)(1 - \rho_0 v)$$

all that has been done, in effect, is to refer the static-pressure-energy term to an elevation datum as well as to a pressure datum and the potential energy term to a density or pressure datum, as well as an elevation datum. The result is to increase the absolute static-pressure potential energy from the value pv that it actually has at elevation Z to the slightly larger value it assumes when referred to the elevation $Z = 0$, where the absolute pressure is $p + Z(g/g_c)\rho_0$. Similarly, the potential head term is decreased from what it would be in a vacuum to the value $Z(g/g_c)(1 - \rho_0 v)$ that it actually has relative to an exterior fluid of density ρ_0. For this reason, the terms $(B + P)v$ and $Z(g/g_c)(1 - \rho_0 v)$ may be thought of, respectively, as buoyant static pressure and potential energies, in recognition of the effect of the external fluid medium. In the same way the energy-difference terms $(P_1 - P_2)v_{avg}$ and $(Z_1 - Z_2)(g/g_c)(1 - \rho_0 v_{Z_{avg}})$ may be termed *buoyant static pressure* and *potential energy differences* to distinguish them from the absolute differences $(p_1 - p_2)v_{avg}$ and $(Z_1 - Z_2)(g/g_c)$ of a system *in vacuo*.

While the use of the term *head* is still retained in speaking of the various terms of Bernoulli's equation, it is better to regard the units of head as comprising foot pounds of force per pound of mass, and not simply as feet of column height. With this approach and the recognition of buoyant static and potential heads as the effective and more realistic forces responsible for flow in a submerged system, the Bernoulli equation may be applied to all types of flow.

When dealing with heating and ventilating problems, most engineering text books and handbooks avoid or at least reduce error in the application of Equation (4) by suggesting that the potential-head-difference term be ignored. It is obvious from Equation (11) that when the system fluid is air at atmospheric pressure and temperature this recommendation does lead to an exact answer. When the density of the system fluid is greater or less than that of the outside fluid, the result is no longer exact although usually much closer to the correct value than it would be were the term $(Z_2 - Z_1)(g/g_c)$ to be included. It is apparent, however, that such a recommendation is entirely empirical and gives better results only because the term $(1 - \rho_0 v_{avg})$ is small, and not because the quantity $(Z_2 - Z_1)(g/g_c)$ is necessarily negligible. As another example, the following problem, given in a familiar text book (*2*), will be considered.

Illustration 2

Problem. Air at 70°F. is heated to 170°F. in a horizontal heater, thereby increasing its velocity from 20 ft./sec. at station 1 to 23.8 ft./sec. at station 2. The absolute pressure at station 1 is given as normal barometer; that at station 2 as 1 in. of water less. After the net heat input to the air in the heater is computed as 24.003 B.t.u./lb. of air, the question is raised as to whether the heat input would have been materially different had the heater exit been arranged vertically with the air leaving the apparatus at an elevation 10 ft. above the inlet.

Solution. The solution given is $(Z_2 - Z_1)/J = 10/778 = 0.0129$ B.t.u./lb. of air, additional heat input required.

However, as the gas passing through the vertical section has a density of 0.0690 lb./cu. ft. as compared with a density of 0.075 lb./cu. ft. for the outside air, it is clear that, other quantities remaining as before, the additional heat input is in reality given by

$$(Z_2 - Z_1)(g/g_c)(1 - \rho_0 v_{avg})/J$$

$$= (10 - 0)(1 - 0.075/0.0690)/778$$

$$= -0.00112 \text{ B.t.u./lb.; i.e.,}$$

the necessary heat input is actually smaller in the vertical than in the horizontal apparatus.

NOTATION

- B = absolute pressure of external buoyant fluid at zero datum of elevation (barometric pressure in the case of air), lb. force/sq. ft.
- D = pipe diameter, ft.
- F = heat input as a result of friction, B.t.u./lb. mass
- f = Fanning friction factor, dimensionless
- g = local acceleration of gravity, ft./sec.^{2}
- g_c = dimensional constant in Newton's law, 32.1740 (lb. mass) (ft.) /(lb. force)(sec.^{2}).
- J = mechanical equivalent of heat, 778 ft. lb. force/B.t.u.
- L = length of a conduit or pipe, ft.
- p = absolute pressure at elevation Z, lb. force/sq. ft.
- p' = absolute pressure at elevation Z referred to zero datum elevation. Buoyant absolute pressure, lb. force/sq. ft., = $(P + B)$
- P = gauge pressure at elevation Z. Pressure difference between system fluid and external fluid at elevation Z, lb. force/sq. ft.
- Q = heat added to the system, B.t.u./lb. mass
- U = internal energy, B.t.u./lb. mass
- V = average linear velocity of fluid flow, ft./sec.
- v = specific volume of system fluid, cu. ft./lb. mass
- v_{avg} = arithmetic average specific volume between stations 1 and 2, cu. ft./lb. mass
- $v_{Z_{avg}}$ = arithmetic average specific volume over section of system containing either an abrupt or gradual change in elevation, cu. ft./lb. mass
- W_s = external work performed by the system, ft. lb. force/lb. mass
- Z = elevation, ft.
- ρ = $1/v$, density, lb. mass/cu. ft.
- ρ_0 = average density of fluid outside a system, between the datum $Z = 0$ and elevations Z_1 and Z_2, lb. mass/cu. ft.

Subscripts

- 1 = upstream section
- 2 = downstream section

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